# Correlated Information and Mechanism Design

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# Introduction

- In most models of private or asymmetric information, possessors of private information receive rents or profits.
- Many applications of the mechanism design paradigm include the assumption that the information held by the players is jointly independently distributed.
- The independence assumption often leads to positive rents accruing to the possessors of private information.
- We provide a condition on the joint distribution of agents' private information which is both necessary and sufficient for reducing the value of this information to zero.

# Motivation

- Consider two players: player 1 and player 2.
- Let $v_1, \ldots, v_n$ be each bidder's set of possible values, and let $P$ be the matrix of bidder 1's conditional probabilities.
- Thus, the $ij$ th entry, $p_{ij}$, of $P$ denotes the probability that bidder 2 has value $v_j$ given that bidder 1 has value $v_i$. Denote by $p_i$, the $i$ th row of $P$.
- Finally, let $\pi_i$ be bidder 1 's expected profit from the Vickrey auction (excluding any participation fees) when his value is $v_i$.
- CM's restriction on $P$ : for all $i = 1, \ldots, n, p_{i.} \notin \text{co}\,\{p_k\}_{k \neq i}$.
- With the conditional distribution satisfying this condition, the auctioneer can extract all of bidder 1's surplus.

# Motivation

- The mechanism to extract all of bidder 1's surplus:
- For each $i = 1, \ldots, n$, there is a hyperplane $x_i \in \mathbb{R}^n$ separating $p_{i\cdot}$ and co $\{p_{k\cdot}\}$ so that $x_i \cdot p_{i\cdot} = 0$ and $x_i \cdot p_{k\cdot} > 0$ for all $k \neq i$.
- Now, for each $m = 1, \ldots, n$ construct the participation fee schedule (for bidder 1) $z_m(j) = \pi_m + \alpha \cdot x_{mj}$, where $\alpha > 0$ will be specified below.
- Thus, if bidder 1 wishes to participate in the Vickrey auction, he must first (knowing his own value) choose a participation fee schedule $z_m(\cdot)$ say, thereby agreeing to pay $z_m(j)$ if player 2 announces a value of $v_j$.
- Since 1's payoff in the auction itself is independent of the participation fee schedule he chooses, he will choose that schedule yielding the lowest expected fee. That is, bidder 1 , given that his value is $i$, will choose $m = 1, \ldots, n$ to minimize $p_{i\cdot} \cdot z_m = \pi_m + \alpha p_{i\cdot} \cdot x_m$.
- Now, since $p_{i\cdot} \cdot x_m > 0$ whenever $m \neq i$ and $p_i \cdot x_i = 0$, we may choose $\alpha > 0$ so that for every $i, p_{i\cdot} \cdot z_m$ is minimized when $m = i$. Hence for every $i = 1, \ldots, n$, if bidder 1 has value $v_i$ he will optimally choose fee schedule $z_i(\cdot)$ and earn an expected surplus of zero.
- Using a similarly constructed set of fee schedules for bidder 2, the auctioneer can in this way extract the full surplus.

# Continuous Type Model

- Now consider the continuum analogue to CM's result. Let $f(s \mid t)$ be the density of $s$ conditional on an agent's type $t \in [0, 1]$, and suppose this agent anticipates profits $\pi(t)$ on average from participation in the Vickery auction.
- The analogous full rent extraction problem for the seller is:
- Construct finitely many participation fee schedules $z_1(\cdot), \ldots, z_N(\cdot)$ so that for all $t \in [0, 1]$

$$\pi(t) = \min_{1 \leqslant n \leqslant N} \int_0^1 z_n(s) f(s \mid t) ds$$

- If such schedules exist, and the agent is risk neutral, then the agent's rents can be extracted.
- However, we can not guarantee the equation is solvable in general.
- Suppose that given $f$, the following were true:

$$\forall \varepsilon > 0, \forall \pi \in C[0, 1], \quad \exists z_1, \ldots, z_N \in C[0, 1] \text{ such that } \forall t \in [0, 1]$$
$$0 \leqslant \pi(t) - \min_{1 \leqslant n \leqslant N} \int_0^1 z_n(s) f(s \mid t) ds < \varepsilon.$$

- Then, regardless of the $\pi$ determined by $f$ as a result of the Vickery auction there is a participation charge which does not induce bidder 1 to refuse to participate given his type (the first inequality in (1.2)) and extracts all but $\varepsilon$ of his rents where $\varepsilon$ is arbitrarily small.

# Continuous Type Model

- Let $R(f)$ denote the set of all such participation charges. Hence,

$$\begin{aligned} R(f) &= \left\{ y : (\exists z \in C[0,1])(\forall t \in [0,1]) y(t) = \int_0^1 z(s) f(s \mid t) ds \right\} \\ &\subseteq C[0,1] \end{aligned}$$

- Note that $R(f)$ is a linear subspace of $C[0,1]$.
- The mechanism designer also has available participation charges that are independent of the agent's report and are not contained in $R(f)$.
- These charges are constructed as follows: Let $N$ be a finite set of indices, and let $z_n$ be a member of $C[0,1]$ for every $n \in N$. Present the agent with a choice of participation charges from $R(f)$. That is, the agent selects $n \in N$, and is then charged $z_n(s)$ when $s$ is realized. The agent of type $t$ will select $n$ minimizing the participation charge:

$$\int_0^1 z_n(s) f(s \mid t) ds.$$

# Continuous Type Model

- These charges are constructed as follows: Let $N$ be a finite set of indices, and let $z_n$ be a member of $C[0,1]$ for every $n \in N$. Present the agent with a choice of participation charges from $R(f)$. That is, the agent selects $n \in N$, and is then charged $z_n(s)$ when $s$ is realized. The agent of type $t$ will select $n$ minimizing the participation charge:

$$\int_0^1 z_n(s) f(s \mid t) ds$$

- If the agent's choice of $n$ is not used in the game to follow, the participation charge given by

$$y(t) = \min_n \int_0^1 z_n(s) f(s \mid t) ds$$

is independent of his reported value in the game to follow. We denote the set of such participation charges by $r(f) \supseteq R(f)$. Thus,

$$r(f) = \left\{ y : (\exists N)(\forall t \in [0,1]) y(t) = \min_{1 \leqslant n \leqslant N} \int_0^1 z_n(s) f(s \mid t) ds \right\} \\ \subset C[0,1].$$

# Continuous Type Model

- The following facts are easily established:
- (2.2) $y_1, y_2 \in r(f) \Rightarrow y_1 + y_2 \in r(f)$,
- (2.3) $y \in r(f), \alpha \geqslant 0 \Rightarrow \alpha y \in r(f)$,
- (2.4) $y_1, \ldots, y_k \in r(f) \Rightarrow \min_{1 \leqslant n \leqslant k} y_n \in r(f)$,
- (2.5) $1, -1 \in r(f)$,
- (2.6) $y_1 \in r(f), y_2 \in R(f) \Rightarrow y_1 - y_2 \in r(f)$.
- The goal is to show that $\bar{r}(f) = C[0,1]$.

# Continuous Type Model

- Before stating the theorem we define for any $\varepsilon > 0, \delta > 0$, and $t_0 \in [0,1]$, the set $U(\varepsilon, \delta, t_0)$ of $(\varepsilon, \delta)$ $u$-shaped functions at $t_0$ as follows: $u \in C[0,1]$ is in $U(\varepsilon, \delta, t_0)$ if and only if
  1. $u(t) \geqslant 0 \quad$ for all $t \in [0,1]$,
  2. $u(t_0) \leqslant \varepsilon, \quad$ and
  3. $u(t) \geqslant 1$ whenever $|t - t_0| > \delta$.
- Note that if $\varepsilon \leqslant \varepsilon_0$ and $\delta \leqslant \delta_0$, then $U(\varepsilon, \delta, t_0) \subseteq U(\varepsilon_0, \delta_0, t_0)$.
- Also note that $U(\varepsilon, \delta, t_0)$ is convex with a nonempty interior.

# Theorem 1

## Theorem (1)

*Suppose* $A \subseteq C[0,1]$ *satisfies*

1. *(2.7)* $x, y \in A \Rightarrow x + y \in A$,
2. *(2.8)* $x \in A, \alpha > 0 \Rightarrow \alpha x \in A$,
3. *(2.9)* $x_1, \ldots, x_n \in \bar{A}, y(t) = \min\{x_1(t), \ldots, x_n(t)\} \Rightarrow y \in \bar{A}$,
4. *(2.10)* $1, -1 \in A$,
5. *(2.11) for all* $\varepsilon, \delta > 0$ *and every* $t \in [0,1]$, $U(\epsilon, \delta, t) \cap \bar{A} \neq \varnothing$. *Then* $\bar{A} = C[0,1]$.

# Theorem 2

- We now present our main result which provides a necessary and sufficient condition (the continuum analogue of that in Crémer and McLean (1988)) for (almost) full rent extraction.

## Theorem (2)

*$\bar{r}(f) = C[0,1]$ if and only if the following condition holds:*

$$\text{(*) For every } t_0 \in [0,1] \text{ and every } \mu \in \Delta[0,1]$$

$$\mu(\{t_0\}) \neq 1 \quad \text{implies that} \quad f(\cdot \mid t_0) \neq \int_0^1 f(\cdot \mid t)\mu(dt)$$

# Necessity

- Proof: We first prove the necessity of $(*)$ for $\bar{r}(f)=C[0,1]$. So, suppose that $\bar{r}(f)=C[0,1]$, and that $t_0\in[0,1]$ and $\mu\in\Delta[0,1]$ satisfy $f(\cdot\mid t_0)=\int_0^1 f(\cdot\mid t)\mu(dt)$. We will show that $\mu(\{t_0\})=1$.
- Let $y(t)=(t-t_0)^2$ for every $t\in[0,1]$. Hence, $y\in C[0,1]=\bar{r}(f)$. There must therefore be a sequence $\{y_n\}_{n=1}^{\infty}$ of functions in $r(f)$ so that $y_n\to y$. Since each $y_n\in r(f)$ we have

$$y_n(t)=\min_{1\leqslant i\leqslant m_n}\left\{w_1^n(t),\ldots,w_{m_n}^n(t)\right\}$$

for every $n$, and every $t\in[0,1]$, where

$$w_i^n(t)=\int_0^1 z_i^n(s)f(s\mid t)ds \quad \text{for some} \quad z_i^n\in C[0,1]$$

- Thus, for each $n$ and every $t\in[0,1]$, $y_n(t)=w_{i(n,t)}^n(t)$ for some $i(n,t)\leqslant m_n$. In particular,

$$\begin{aligned}
y_n(t_0) &= w_{i(n,t_0)}^n(t_0)\\
&= \int_0^1 z_{i(n,t_0)}^n(s)f(s\mid t_0)ds\\
&= \int_0^1\int_0^1 z_{i(n,t_0)}^n(s)f(s\mid t)\mu(dt)ds\\
&= \int_0^1 w_{i(n,t_0)}^n(t)\mu(dt)
\end{aligned}$$

# Necessity

- Since $y_n(t_0) \to y(t_0) = 0$, this implies that the last integral converges to zero. Now, by definition, $y_n(t) \leqslant w^n_{i(n,t_0)}(t)$ so that (since $\mu \in \Delta[0,1]$)

$$\int_0^1 y_n(t)\mu(dt) \leqslant \int_0^1 w^n_{i(n,t_0)}(t)\mu(dt) \to 0$$

- Hence, $0 \geqslant \int_0^1 y(t)\mu(dt) = \int_0^1 (t - t_0)^2 \mu(dt)$, so that $\mu(\{t_0\}) = 1$.

# Sufficiency

- We turn now to sufficiency, and proceed by way of contradiction. Suppose that $\bar{r}(f) \neq C[0,1]$ and that (*) holds. Since hypotheses (2.7)-(2.10) of Theorem 1 are satisfied when $\bar{A} = r(f)$, it must be the case (by Theorem 1) that (2.11) fails when $\bar{A}$ is replaced by $\bar{r}(f)$.
- Thus, there exist $\varepsilon_0, \delta_0 > 0$, and $t_0 \in [0,1]$ such that $U(\varepsilon_0, \delta_0, t_0) \cap \bar{r}(f) = \varnothing$. Since $\bar{R}(f) \subseteq \bar{r}(f)$ we have a fortiori that $U(\varepsilon_0, \delta_0, t_0) \cap \bar{R}(f) = \varnothing$.
- Now, $\bar{R}(f)$ is convex (being a linear subspace) and as previously noted, $U(\varepsilon_0, \delta_0, t_0)$ is convex and has a nonempty interior. So, by the separating hyperplane theorem (Dunford and Schwartz (1958, 1988; Theorem 8, p. 417), there is a continuous linear functional on $C[0,1]$ separating $\bar{R}(f)$ and $U(\varepsilon_0, \delta_0, t_0)$.
- Equivalently, by the Riesz representation theorem (Dunford and Schwartz (1958, 1988; Theorem 3, p. 265)), there is a regular, countably additive, signed measure $\mu \neq 0$ on the Borel subsets of $[0,1]$ and a constant $c \in \mathbb{R}$ such that

$$\begin{array}{lll} \int_0^1 x(t)\mu(dt) \leqslant c & \text{for all} & x \in \bar{R}(f), \text{ and} \\ \int_0^1 x(t)\mu(dt) \geqslant c & \text{for all} & x \in U(\varepsilon_0, \delta_0, t_0) \end{array}$$

- Since $\bar{R}(f)$ is a linear subspace, we must therefore have $\int_0^1 x(t)\mu(dt) = 0$ for every $x \in \bar{R}(f)$. (Otherwise there is an $x_0 \in \bar{R}(f)$ with $\int_0^1 x_0(t)\mu(dt) \neq 0$, and a suitable choice of $\alpha \in \mathbb{R}$ yields $\int_0^1 \alpha x_0(t)\mu(dt) > c$, violating (2.12) since $\alpha x_0 \in \bar{R}(f)$.) Hence, $c$ can be taken to be zero without loss of generality.

# Sufficiency

- Combining (2.12) and the definition of $R(f)$ we then have

$$\int_0^1 \left\{ \int_0^1 z(s) f(s \mid t) ds \right\} \mu(dt) = 0 \quad \text{for every} \quad z \in C[0,1]$$

- By Fubini's theorem, this is equivalent to

$$\int_0^1 z(s) \left[ \int_0^1 f(s \mid t) \mu(dt) \right] ds = 0 \quad \text{for every} \quad z \in C[0,1]$$

- Hence, the continuous function of $s$ in square brackets is identically zero. That is

$$\int_0^1 f(\cdot \mid t) \mu(dt) = 0$$

- By the Jordan decomposition theorem (Cohn (1980, Corollary 4.1.5, p. 125)), we may write $\mu$ as the difference between two positive measures $\mu^+$and $\mu^-$at least one of which is finite. Furthermore, there are disjoint Borel subsets of $[0,1]$, $A^+$and $A^-$, such that $\mu^+(A^-) = \mu^-(A^+) = 0$, and $A^+ \cup A^- = [0,1]$. Thus (2.14) becomes (2.15)

$$\int_{A^+} f(\cdot \mid t) \mu^+(dt) = \int_{A^-} f(\cdot \mid t) \mu^-(dt).$$

# Sufficiency

- Regarding both sides of (2.15) as functions of $s \in [0, 1]$, integrating over $s$ (with respect to Lebesgue measure) and using Fubini's theorem yields $\int_{A^+} d\mu^+ = \int_{A^-} d\mu^- = 1$, where the second equality is without loss of generality. Hence, both $\mu^+$and $\mu^-$are in $\Delta[0, 1]$.
- Combining (*), (2.15), and the fact that $\mu \neq 0$, yields that neither $\mu^+$nor $\mu^-$ is a point mass on $t_0$. In particular, since $\mu^- \in \Delta[0, 1]$ is regular (see Billingsley (1968, Theorem 1.1)), there is a closed subset $B$ of $A^-$, and a $\delta \in (0, \delta_0]$ such that $B \cap (t_0 - \delta, t_0 + \delta) = \varnothing$ and $\mu^-(B) > 0$. Choose $K > 1/\mu^-(B) \geqslant 1$, and define the step function $x$ on $[0, 1]$ as follows:

$$x(t) = \begin{cases} 0, & \text{if } t \in (t_0 - \delta, t_0 + \delta) \\ K, & \text{if } t \in B \\ 1, & \text{otherwise} \end{cases}$$

- Hence, $\int_0^1 x(t)\mu(dt) \leqslant 1 - K\mu^-(B) < 0$.

# Sufficiency

- Now, using Theorem 1.2 of Billingsley, it is straightforward to construct a sequence of continuous functions $\{x_n\}_{n=1}^{\infty}$ on $[0,1]$ such that for every $n$,
- (i) $x_n(t) \geqslant 1$, for every $t \notin (t_0 - \delta, t_0 + \delta)$,
- (ii) $x_n(t) \geqslant 0$, for every $t \in [0,1]$,
- (iii) $x_n(t_0) = 0$,
- (iv) for every $t \in [0,1]$, $\quad x_n(t) \to x(t)$,
- (v) for every $t \in [0,1]$, $\quad x_n(t) \leqslant K$.
- By (i)-(iii) $x_n \in U(\varepsilon_0, \delta, t_0) \subseteq U(\varepsilon_0, \delta_0, t_0)$ (since $\delta \leqslant \delta_0$ ), for every $n$. And by (ii), (iv), (v), and Lebesgue's dominated convergence theorem, $\int_0^1 x_n(t)\mu(dt) \to \int_0^1 x(t)\mu(dt) < 0$. Thus for $n$ large enough, $\int_0^1 x_n(t)\mu(dt) < 0$, contradicting (2.13).

# Condition

- Note that if for every $t_0 \in [0, 1]$, there is an $x_{t_0} \in \bar{r}(f)$ taking a minimum uniquely at $t_0$, then setting $y(t) = x_{t_0}(t) - x_{t_0}(t_0)$ in the proof of necessity above is enough to show that $(^*)$ holds and hence (by sufficiency) that $\bar{r}(f) = C[0, 1]$.
- This observation is at the heart of the three corollaries which follow. Like Theorem 1, Theorem 2 also holds if $[0, 1]$ is replaced by any compact metric space.

## Corollary (1)

*Suppose $x \in R(f), y \in r(f)$ satisfy*

$$(\forall t) x'(t) > 0$$
$$(\forall t) y'(t)/x'(t) \text{ is strictly increasing in } t$$

*Then $\bar{r}(f) = C[0, 1]$.*

# Condition

## Corollary (1)

*Suppose* $x \in R(f), y \in r(f)$ *satisfy*

$$(\forall t)x'(t) > 0$$
$$(\forall t)y'(t)/x'(t) \text{ is strictly increasing in } t$$

*Then* $\bar{r}(f) = C[0,1]$.

- Proof: As noted in Remark 2, we need only find for each $t_0 \in [0,1]$ a function in $\bar{r}(f)$ taking a minimum uniquely at $t_0$. Let

$$q(t) = y(t) - \frac{y'(t_0)}{x'(t_0)}x(t)$$

- By (2.6), $q \in r(f)$. Moreover

$$q'(t) = y'(t) - \frac{y'(t_0)}{x'(t_0)}x'(t) \gtreqless 0 \quad \text{as} \quad \frac{y'(t)}{x'(t)} \gtrless \frac{y'(t_0)}{x'(t_0)} \quad \text{as} \quad t \gtreqless t_0$$

- Thus $q(t)$ achieves a minimum uniquely at $t = t_0$.

# Condition

- By Corollary 1, it is straightforward to show that combined with first order stochastic dominance, a sufficient condition for $\bar{r}(f) = C[0,1]$ is that $E\left[s^2 \mid E(s \mid t) = \mu\right]$ be a strictly convex function of $\mu$. The following example illustrates this.
- Example 1: $f(s \mid t) = ts^{t-1}$ and $\mu = E(s \mid t) = \frac{t}{t+1}$ implies that $t = \frac{\mu}{1-\mu}$. Also, $E\left(s^2 \mid t\right) = \frac{t}{t+2}$, so that $E\left\{s^2 \mid Es = \mu\right\} = \frac{\mu}{2-\mu}$, a convex function of $\mu \in [0, \frac{1}{2}]$. Thus, letting $x(t) = E(s \mid t) = \frac{t}{t+1}$ and $y(t) = E\left(s^2 \mid t\right) = \frac{t}{t+2}$, we have that $\frac{y'(t)}{x'(t)} = 2(\frac{t+1}{t+2})^2$ is increasing in $t$.
- Since by first order stochastic dominance $x'(t) > 0$, Corollary 1 can be directly applied to conclude that $\bar{r}(f) = C[0,1]$. In general, with first order stochastic dominance and $E\left[s^2 \mid E(s \mid t) = \mu\right]$ a convex function of $\mu, x(t) = E(s \mid t)$ and $y(t) = E\left(s^2 \mid t\right)$ will satisfy the hypotheses of Corollary 1. Furthermore, in this case the participation fee schedules $z_n(s)$, can be chosen to be quadratic in $s$. These conditions are satisfied for many common distributions, in particular those with mean and variance increasing in $t$.

# Condition

- The lemma to follow establishes a useful equivalence for placing functions in $\bar{R}(f)$. $[A]$ denotes the linear span of $A$.

## Lemma (1)

$\bar{R}(f) = \overline{[\{f(s \mid \cdot) : 0 \leqslant s \leqslant 1\}]}$.

- Proof of Lemma 1: Fix $s_0 \in [0,1]$ and $\varepsilon > 0$. For $s \in [0,1]$, let

$$z(s) = \begin{cases} 1/2\alpha & \text{if } s_0 - \alpha \leqslant s \leqslant s_0 + \alpha \\ 0 & \text{otherwise} \end{cases}$$

and choose $\alpha$ so that

$$|s - s_0| < \alpha \Rightarrow |f(s \mid t) - f(s_0 \mid t)| < \varepsilon$$

(this is feasible since $f$ is continuous on a compact set, and hence uniformly continuous). Then

$$\begin{aligned} & \left| \int_0^1 z(s) f(s \mid t) ds - f(s_0 \mid t) \right| \\ & = \left| \int_{s_0-\alpha}^{s_0+\alpha} \frac{1}{2\alpha} f(s \mid t) ds - f(s_0 \mid t) \right| = \frac{1}{2\alpha} \left| \int_{s_0-\alpha}^{s_0+\alpha} \left( f(s \mid t) - f(s_0 \mid t) \right) ds \right| \\ & \leqslant \frac{1}{2\alpha} \int_{s_0-\alpha}^{s_0+\alpha} |f(s \mid t) - f(s_0 \mid t)| \, ds \leqslant \frac{1}{2\alpha} \int_{s_0-\alpha}^{s_0+\alpha} \varepsilon ds = \varepsilon \end{aligned}$$

- Thus, for $s_0 \in [0,1]$, $f(s_0 \mid \cdot) \in \bar{R}(f)$. Since $\bar{R}(f)$ is closed under linear combinations, we have established one inclusion.

# Condition

- Since $z, f$ are continuous, $\forall \varepsilon > 0 \exists s_1, \ldots, s_k$ such that for all $t \in [0,1]$,

$$\left| \int_0^1 z(s) f(s \mid t) ds - 1/k \sum_{i=1}^{k} z(s_i) f(s_i \mid t) \right| < \varepsilon$$

Thus $\int_0^1 z(s) f(s, \cdot) ds \in \overline{[\{f(s \mid \cdot) \mid s \in [0,1]\}]}$ as desired.

## Corollary (2)

*Suppose that (2.16)* $\quad (\forall t)(\exists s)(\forall t' \neq t) \quad f(s \mid t) > f(s \mid t')$. *Then* $\bar{r}(f) = C[0,1]$.

- Proof: By Lemma 1, $-f(s \mid \cdot) \in \bar{r}(f)$. By (2.16), for each $t$, there exists an $s$ with $-f(s \mid \cdot)$ taking a minimum uniquely at $t$. In light of Remark 2, $\bar{r}(f) = C[0,1]$.

# Condition

- The final result of this section provides further conditions for rent extraction which, in some instances, are simple to verify.

## Corollary (3)

*Suppose there exists a set* $S \subseteq [0,1]$ *such that*

1. *(2.17)* $(\forall s \in S) \quad f(s \mid t)$ *is strictly concave in* $t$*, and*
2. *(2.18)* $(\forall t_0, t_1 \in [0,1])\,(\exists s \in S) \quad f\,(s \mid t_0) \geqslant f\,(s \mid t_1)$.

*Then* $\bar{r}(f) = C[0,1]$.

- Proof: Suppose (2.17) and (2.18) hold, but (*) fails. Then there exists a $t_0 \in [0,1]$ and $\mu \in \Delta[0,1]$ not a point mass on $t_0$, with $f\,(\cdot \mid t_0) = \int f(\cdot \mid t)\mu(dt)$. Define $t_1 = \int t\mu(dt)$. Since $f(s \mid t)$ is strictly concave in $t$, for all $s \in S$, we have, by Jensen's inequality,

$$(\forall s \in S) \quad f\,(s \mid t_0) = \int f(s \mid t)\mu(dt) < f\,(s \mid t_1)$$

which contradicts (2.18).

# Condition

- We now show by example that the combination of first order stochastic dominance and affiliation is not sufficient to guarantee $\bar{r}(f) = C[0,1]$. As the example illustrates, the combination of these properties admits an $f$ with $R(f)$ comprised of only linear functions and no $u$-shaped functions.
- Example 2: $f(s \mid t) = 1 + (2s-1)t$. Note $\int_0^1 f(s \mid t)ds = 1 + \left(s^2 - s\right) t\big|_0^1 = 1$, and $f(s \mid t) \geqslant 1 - t \geqslant 0$, so $f$ is an admissible conditional density.

$$F(s \mid t) = \int_0^s f(u \mid t)du = s + t\left(s^2 - s\right)$$
$$F_t(s \mid t) = s^2 - s < 0 \text{ for } s \in (0,1)$$

  so $F$ satisfies strict first order stochastic dominance.
- Also,

$$\frac{\partial^2}{\partial s \partial t} \log f(s \mid t) = \frac{\partial}{\partial s} \frac{2s-1}{1+(2s-1)t} > 0$$

  so $f$ is affiliated (see Milgrom and Weber (1982)). Equivalently, $f$ has the monotone likelihood ratio property.
- Finally, $R(f) = [\{1, t\}]$, the set of linear functions. (1 indicates the constant function, $t$ the identity.) It is easily seen that $\bar{r}(f)$ is then the set of concave functions, and is thus a strict subset of $C[0,1]$.

# Unbounded Support

- We mention briefly that the results of this section can be extended in a straightforward manner to the unbounded support case. This may require allowing players to choose from among countably (rather than finitely) many participation charges, so in what immediately follows $r(f)$ is:

$$r(f) = \left\{ y(t) \in C(\mathbb{R}) \mid y(t) = \min_{n \geqslant 1} \int z_n(s) f(s \mid t) ds \right.$$

  for some countable subset $\{z_n\}_{n=1}^{\infty}$ of $C(\mathbb{R})\}$, where $C(\mathbb{R})$ denotes the set of bounded continuous functions on $\mathbb{R}$.

- Application: consider a principal designing a contract for a risk neutral agent possessing private information $t \in [0, 1]$. The principal knows he can receive signal $s$ correlated to $t$, sometime in the future. What is the value of $s$?
- Consider the full information gains from trade $G$, and the solution to the informationally constrained contract design problem, which gives the principal profits of $G'$. We have shown that if an efficient mechanism exists, then, for many densities, the value of the correlated information is $G - G'$. This follows since the principal can set up a mechanism which is full-information efficient, producing rents $G$, and then extract those rents via a participation charge $z_n(s)$. That is, the principal "sells the agency" to the agent for $z_n(s)$. We believe that, in many economic problems, the presence of correlated information is natural, and destroys the "inefficiencies resulting from private information" so often cited in the literature.

# Applications

- Bargaining mechanism: Consider a buyer with value $t$, known only to himself, of an item and a potential seller, who privately observes his own opportunity cost of sale, $s$. It is common knowledge that $s$ and $t$ were drawn from a joint density $g(s,t)$ with support $[0,1]^2$. Both buyer and seller are risk neutral.
- Let $h(s \mid t)$ be the conditional density of the seller's value given that the buyer's value is $t$ and let $k(t \mid s)$ be the conditional density of the buyer's value given the seller's value is $s$, and suppose that both $h$ and $k$ satisfy (*) (ruling out independence, in particular).
- Then, letting $\pi^\sigma, \pi^\beta$ denote the seller's, buyer's rent function (a function of their respective value of the good) respectively obtained from participation in the game defined by the pre-mechanism, we have, for $(s,t) \in [0,1]^2$,

$$\pi^\sigma(s) \equiv \int_0^1 (t-s)k(t \mid s)dt,$$
$$\pi^\beta(t) \equiv \int_0^1 (t-s)h(s \mid t)ds.$$

- Now, by assumption $\bar{r}(k) = \bar{r}(h) = C[0,1]$. Hence, given any $\varepsilon > 0$ there exist finite sets of participation fee schedules $\left\{z_n^\beta\right\}_{n\in N_\beta}, \{z_n^\sigma\}_{n\in N_\sigma}$, one for the buyer and one for the seller, such that for all $(s,t) \in [0,1]^2$,

$$0 \leqslant \pi^\sigma(s) - \min_{n\in N_\sigma} \int_0^1 z_n^\sigma(t)k(t \mid s)dt < \varepsilon$$

and

$$0 \leqslant \pi^\beta(t) - \min_{n\in N_\beta} \int_0^1 z_n^\beta(s)h(s \mid t)ds < \varepsilon$$

# Applications

- Hence, the budget balancer's net expected revenue becomes:

$$
\begin{aligned}
R &\equiv \int_0^1 \int_0^1 c^\sigma(s) g(s,t)\,ds\,dt + \int_0^1 \int_0^1 c^\beta(t) g(s,t)\,dt\,ds - G \\
&\geqslant (G-\varepsilon) + (G-\varepsilon) - G \\
&= G - 2\varepsilon
\end{aligned}
$$

- As before, $g(s,t)$ is the joint density between the buyer's and seller's valuation of the good. Let

$$f(s \mid t) = g(s,t) / \int_0^1 g(u,t)\,du$$

so that $f$ is the conditional density, and let

$$F(s \mid t) = \int_0^s f(u \mid t)\,du$$

be the distribution function of $s$, conditional on $t$. Let $F_2(s \mid t) = \partial/\partial t F(s \mid t)$.

# Applications

- By taking advantage of our explicit description of this mechanism design environment, we get the following result:

## Theorem (3)

*Suppose* $\forall (s,t) \in (0,1)^2$,

$$F_2(s \mid t) < 0$$
$$\frac{\partial}{\partial_t}\left[t + \frac{F(s \mid t)}{F_2(s \mid t)}\right] \geqslant 0.$$

*Then there exists an efficient trading mechanism giving all of the rents to the seller.*

# Conclusion

- We have examined the robustness of mechanism design solutions when independence of information does not hold.
- We found that private information is often worthless; it does not lead to rents for its possessors in a variety of contexts.

Thanks!