Efficient dynamic mechanisms in environments with interdependent valuations: The role of contingent transfers

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Introduction

- This paper studies efficient mechanism design in dynamic allocation problems with interdependent valuations.
- Two features: interdependent valuation & dynamic.

Model

- We consider a dynamic interdependent valuation environment with $N (N \geq 2)$ agents.
- Time is discrete, indexed by $t \in \{1, 2, \ldots, T\}$, where $T \leq \infty$.
- In each period t, each agent $i \in \{1, 2, \ldots, N\}$ privately observes a payoff-relevant signal $\theta_t^i \in \Theta_t^i$, where Θ_t^i is a finite set.
- The signal space in period t is $\Theta_t = \prod_{i=1}^N \Theta_t^i$ with a generic element $\theta_t = \left(\theta_t^1, \ldots, \theta_t^N\right)$.
- For each i and t, denote the private information held by agents other than i in period t by $\theta_t^{-i} = \left(\theta_t^1, \ldots, \theta_t^{i-1}, \theta_t^{i+1}, \ldots, \theta_t^N\right) \in \prod_{j \neq i} \Theta_t^j$.
- Given sequences of signals $\{\theta_t\}_{t=1}^T$, allocations $\{a_t\}_{t=1}^T$, and monetary transfers $\left\{p_t^1, \ldots, p_t^N\right\}_{t=1}^T$, the total payoff of each agent i is

$$\sum_{t=1}^{T} \delta^{t-1} \left[u^i \left(a_t, \theta_t\right) - p_t^i \right].$$

- The agent's private signals evolve over time following a Markov decision process. In the initial period, the signal profile θ_1 is drawn from a prior probability $\mu_1 \in \Delta\left(\Theta_1\right)$. In each period $t > 1$, the distribution of current signal profile θ_t is determined by the realized signal profile θ_{t-1} and the allocation decision a_{t-1} in the previous period, represented by a transition probability $\mu_t : A_{t-1} \times \Theta_{t-1} \to \Delta\left(\Theta_t\right)$.

Efficiency and Mechanisms

- A socially efficient allocation rule is a sequence of functions $\{a_t^* : \Theta_t \to A_t\}_{t=1}^T$ that solves the social program

$$\max_{\{a_t\}_{t=1}^T} \mathbb{E}\left[\sum_{t=1}^{T} \delta^{t-1} \sum_{i=1}^{N} u^i(a_t, \theta_t)\right],$$

where the expectation is taken with respect to the processes $\{\theta_t\}$ and $\{a_t\}$. Since the flow utility depends only on the current signal profile, which is assumed to be Markov, the social program can be written in the recursive form: for each $t \in \{1, 2, \ldots, T\}$,

$$W_t(\theta_t) = \max_{a_t \in A} \sum_{i=1}^{N} u^i(a_t, \theta_t) + \delta \mathbb{E}\left[W_{t+1}(\theta_{t+1}) \mid a_t, \theta_t\right]$$

where $W_t(\theta_t)$ is the social surplus starting from period t given the realized signal profile θ_t, and $W_{T+1} \equiv 0$. By the principle of optimality, a_t^* solves the social program if and only if it is a solution to this recursive problem.

Efficiency and Mechanisms

- We focus on truthful equilibria of direct public mechanisms that implement the socially efficient allocations $\{a_t^*\}_{t=1}^T$.
- In a direct public mechanism, in each period t, each agent i is asked to make a public report $r_t^i \in \Theta_t^i$ of her current private signal θ_t^i.
- Then a public allocation decision a_t and a transfer p_t^i for each agent i are made as functions of the current report profile $r_t = \left(r_t^i\right)_{i=1}^N$ and the period-t public history h_t.
- The period-t public history contains all reports and allocations up to period $t-1$, i.e.,

$$h_t = \left(r_1, a_1, r_2, a_2, \ldots, r_{t-1}, a_{t-1}\right).$$

- Let H_t denote the set of possible period-t public histories. Formally, an efficient direct revelation mechanism $\Gamma = \{\Theta_t, a_t^*, p_t\}_{t=1}^T$ consists of
  1. Θ_t as the message space in each period t,
  2. a sequence of allocation rules $a_t^* : \Theta_t \to A$,
  3. a sequence of monetary transfers $p_t : H_t \times \Theta_t \to \mathbb{R}^N$.

Efficiency and Mechanisms

- The period-t private history h_t^i of each agent i contains the period-t public history and the sequence of her realized private signals until period t, i.e.,

$$h_t^i = \left(r_1, a_1, \theta_1^i, r_2, a_2, \theta_2^i, \ldots, r_{t-1}, a_{t-1}, \theta_{t-1}^i, \theta_t^i\right)$$

- Let H_t^i denote the set of agent i's possible period-t private histories. With a slight abuse of notation, a strategy for agent i is a sequence of mappings $r^i = \left\{r_t^i\right\}_{t=1}^T$, where $r_t^i : H_t^i \to \Theta_t^i$, that assign a report to each of her period-t private histories. A strategy for agent i is truthful if it always reports agent i 's private signal θ_t^i truthfully in each period t, regardless of her private history.
- Given a mechanism $\Gamma = \{\Theta_t, a_t^*, p_t\}_{t=1}^T$ and a strategy profile $r = \left\{r^i\right\}_{i=1}^N$, agent i 's expected discounted payoff is

$$\mathbb{E} \sum_{t=1}^{T} \delta^{t-1} \left[u^i\left(a_t^*\left(r_t\right), \theta_t\right) - p_t^i\left(h_t, r_t\right)\right].$$

Efficiency and Mechanisms

- We say that the mechanism is periodic ex-post incentive compatible or, equivalently, the truthful strategy profile is a periodic ex-post equilibrium if for each agent and in each period, truth-telling is always a best response regardless of the private history and the current signals of other agents, given that other agents adopt truthful strategies.
- Formally, let $V_t^i\left(h_t^i\right)$ be agent i 's continuation payoff given period-t private history, given that other agents report truthfully. That is,

$$V_t^i\left(h_t^i\right)=\max_{r_t^i\in\Theta_t^i}\mathbb{E}\left[u^i\left(a_t^*\left(r_t^i,\theta_t^{-i}\right),\theta_t\right)-p_t^i\left(h_t,r_t^i,\theta_t^{-i}\right)+\delta V_{t+1}^i\left(h_{t+1}^i\right)\right].$$

- The efficient mechanism is periodic ex-post incentive compatible if for each i,t, and h_t^i,

$$\theta_t^i\in\arg\max_{r_t^i\in\Theta_t^i}u^i\left(a_t^*\left(r_t^i,\theta_t^{-i}\right),\theta_t\right)-p_t^i\left(h_t,r_t^i,\theta_t^{-i}\right)+\delta\mathbb{E}\left[V_{t+1}^i\left(h_{t+1}^i\right)\mid a_t^*\left(r_t^i,\theta_t^{-i}\right),\theta_t\right]$$

- Finally, the Vickrey-Clarke-Groves (VCG) mechanism is an efficient mechanism $\Gamma=\{\Theta_t,a_t^*,p_t\}_{t=1}^T$ under which each agent i's continuation payoff is equal to the continuation social surplus net of a term that is independent of her current and future reports, i.e., for each i and t, there is a function $W_t^{-i}(\cdot)$ such that

$$V_t^i\left(h_t^i;h_t,\theta_t^{-i}\right)=W_t\left(\theta_t\right)-W_t^{-i}\left(\theta_1^{-i},\ldots,\theta_t^{-i}\right)$$

for all h_t^i,h_t, and θ_t^{-i}.

Main Results

- In this section, we construct periodic ex post incentive compatible efficient dynamic mechanisms under general transition dynamics.
- Theorem 3.1 shows that under a generic intertemporal correlation condition and some restrictions on utility functions and signal spaces in the last period, such a dynamic mechanism always exists.
- In particular, we show that in each period t the correlation between θ_t^i and θ_{t+1}^{-i} can be used to construct history-dependent transfers such that agent i 's incentive is aligned with the social incentive.
- In Theorem 3.2, we show that a slightly stronger intertemporal correlation condition ensures dynamic efficiency with a sequence of "VCG-type" transfers.
- We make the following assumptions on the utility functions and the evolution of private information.

Assumption (1 Bounded payoffs)

For each agent i,

$$\max_{(a_t,\theta_t)_{t\geq 1}} \sum_{t=1}^{T} \delta^{t-1} \left| u^i\left(a_t, \theta_t\right) \right| < \infty$$

Main Results

Assumption (2 Convex independence)

For each $1 \leq t \leq T, i \in N, a_t \in A_t$, *and* $\theta_t^{-i} \in \Theta_t^{-i}$, *no column of the matrix*

$$M_{t+1}^{-i}\left(a_t, \theta_t^{-i}\right) \equiv \left[\mu_{t+1}^{-i}\left(\theta_{t+1}^{-i} \mid a_t, \theta_t^i, \theta_t^{-i}\right)\right]_{\left|\Theta_{t+1}^{-i}\right| \times \left|\Theta_t^i\right|}$$

is a convex combination of other columns, i.e., for each θ_t^i,

$$\mu_{t+1}^{-i}\left(\cdot \mid a_t, \theta_t^i, \theta_t^{-i}\right) \notin \operatorname{Conv}\left\{\mu_{t+1}^{-i}\left(\cdot \mid a_t, \tilde{\theta}_t^i, \theta_t^{-i}\right)\right\}_{\tilde{\theta}_t^i \in \Theta_t^i \setminus \{\theta_t^i\}}$$

where $\operatorname{Conv}\left\{\mu_{t+1}^{-i}\left(\cdot \mid a_t, \tilde{\theta}_t^i, \theta_t^{-i}\right)\right\}_{\tilde{\theta}_t^i \in \Theta_t^i \setminus \{\theta_t^i\}}$ *is the convex hull generated by the set of vectors* $\left\{\mu_{t+1}^{-i}\left(\cdot \mid a_t, \tilde{\theta}_t^i, \theta_t^{-i}\right)\right\}_{\tilde{\theta}_t^i \in \Theta_t^i \setminus \{\theta_t^i\}}$. *Moreover, the transition probabilities satisfy*

$$\inf_{t, i, a_t, \theta_t^{-i}, \theta_t^i} \operatorname{dist}_2\left(\mu_{t+1}^{-i}\left(\cdot \mid a_t, \theta_t^i, \theta_t^{-i}\right), \operatorname{Conv}\left\{\mu_{t+1}^{-i}\left(\cdot \mid a_t, \tilde{\theta}_t^i, \theta_t^{-i}\right)\right\}_{\tilde{\theta}_t^i \in \Theta_t^i \setminus \{\theta_t^i\}}\right) > 0.$$

Main Results

Assumption (3 Spanning condition)

For each $1 \leq t \leq T, i \in N, a_t \in A_t$, *and* $\theta_t^{-i} \in \Theta_t^{-i}$, *the column vectors of the matrix*

$$M_{t+1}^{-i}\left(a_t, \theta_t^{-i}\right) \equiv \left[\mu_{t+1}^{-i}\left(\theta_{t+1}^{-i} \mid a_t, \theta_t^i, \theta_t^{-i}\right)\right]_{\left|\Theta_{t+1}^{-i}\right| \times \left|\Theta_t^i\right|}$$

are linearly independent, i.e., there does not exist a collection of real numbers $\left\{\eta^i\left(\theta_t^i\right)\right\}_{\theta_t^i \in \Theta_t^i}$, *which are not all equal to zero, such that*

$$\sum_{\theta_t^i \in \Theta_t^i} \eta^i\left(\theta_t^i\right) \mu_{t+1}^{-i}\left(\theta_{t+1}^{-i} \mid a_t, \theta_t^i, \theta_t^{-i}\right) = 0$$

for all $\theta_{t+1}^{-i} \in \Theta_{t+1}^{-i}$. *Moreover, if* $T = \infty$, *then there exist* $\bar{D} \in \mathbb{R}_+$ *and* $\bar{T} \in \mathbb{N}_+$ *such that for any* $t \geq \bar{T}$, *any* i, a_t *and* θ_t^{-i}, *the norm of the pseudo-inverse of the matrix* $M_{t+1}^{-i}\left(a_t, \theta_t^{-i}\right)$ *satisfies*

$$\left\|\left(M_{t+1}^{-i}\left(a_t, \theta_t^{-i}\right)\right)^+\right\| \leq \bar{D}.$$

Main Results

- In the finite-horizon case ($T < \infty$), we also impose the following ex post incentive compatibility assumption on the allocation rule a_T^*.

Assumption (4. Ex post incentive compatibility in period T)

If $T < \infty$, then the efficient allocation in period T, a_T^*, is ex post incentive compatible.

Theorem (3.1)

Under Assumptions 1, 2, and 4, there exists a sequence of transfers

$$p_{t+1}^i : \Theta_{t+1}^{-i} \times \Theta_t^i \times A_t \times \Theta_t^{-i} \to \mathbb{R} \quad \forall i, t < T$$

such that the efficient dynamic mechanism $\{a_t^*, p_t\}$ is periodic ex post incentive compatible.

Main Results

Lemma (A.3)

If for each i and t, there exists a transfer function $p^i_{t+1}\left(\theta^{-i}_{t+1}, r^i_t; a_t, \theta^{-i}_t\right)$ that satisfies the three conditions (i) for each a_t, θ^{-i}_t and θ^i_t,

$$-\sum_{j\neq i} u^j\left(a_t, \theta^i_t, \theta^{-i}_t\right) = \delta \sum_{\theta_{t+1}\in\Theta_{t+1}} p^i_{t+1}\left(\theta^{-i}_{t+1}, \theta^i_t; a_t, \theta^{-i}_t\right)\mu\left(\theta_{t+1} \mid a_t, \theta^i_t, \theta^{-i}_t\right),$$

(ii) for each $a_t, \theta^{-i}_t, \theta^i_t$ and r^i_t,

$$\begin{aligned}&\sum_{\theta_{t+1}\in\Theta_{t+1}} p^i_{t+1}\left(\theta^{-i}_{t+1}, \theta^i_t; a_t, \theta^{-i}_t\right)\mu\left(\theta_{t+1} \mid a_t, \theta^i_t, \theta^{-i}_t\right)\\ &\leq \sum_{\theta_{t+1}\in\Theta_{t+1}} p^i_{t+1}\left(\theta^{-i}_{t+1}, r^i_t; a_t, \theta^{-i}_t\right)\mu\left(\theta_{t+1} \mid a_t, \theta^i_t, \theta^{-i}_t\right)\end{aligned}$$

(iii) there exists $\epsilon \in \mathbb{R}_+$ such that for any $t \geq \tilde{T}$,

$$\max\left|p^i_{t+1}\left(\theta^{-i}_{t+1}, r^i_t; a_t, \theta^{-i}_t\right)\right| \leq \frac{1}{\delta}\left(1+\frac{4}{\epsilon}\right)\cdot \max_{a_t,\theta_t}\left|\sum_{j\neq i} u^j\left(a_t, \theta_t\right)\right|,$$

then the dynamic efficient allocation $\left\{a^*_t\right\}$ can be implemented in a periodic ex post equilibrium.

Main Results

- Proof. First note that under condition (iii) of this lemma, agent i 's discounted payoffs are always well defined under the mechanism $\left\{a_t^*, \{p_t^i\}_{i=1}^N\right\}_{t\geq 1}$. To see this, for any sequence $(a_t, \theta_t)_{t\geq 1}$, we have

$$\begin{aligned}
&\sum_{t=1}^{\infty} \delta^{t-1} \left| u^i\left(a_t, \theta_t\right) - p_t^i\left(\theta_t^{-i}; a_{t-1}, \theta_{t-1}^{-i}\right)\right| \\
&\quad = \sum_{t=1}^{\tilde{T}} \delta^{t-1} \left| u^i\left(a_t, \theta_t\right) - p_t^i\left(\theta_t^{-i}; a_{t-1}, \theta_{t-1}^{-i}\right)\right| \\
&\quad + \sum_{t=\tilde{T}}^{\infty} \delta^{t} \left| u^i\left(a_{t+1}, \theta_{t+1}\right) - p_{t+1}^i\left(\theta_{t+1}^{-i}; a_t, \theta_t^{-i}\right)\right| \\
&\leq L^i + \sum_{t=\tilde{T}}^{\infty} \delta^t \left[\left|u^i\left(a_{t+1}, \theta_{t+1}\right)\right| + \frac{1}{\delta}\left(1+\frac{4}{\epsilon}\right)\cdot \max_{a_t, \theta_t}\left|\sum_{j\neq i} u^j\left(a_t, \theta_t\right)\right|\right] \\
&\leq L^i + \frac{1}{\delta}\left(1+\frac{4}{\epsilon}\right)\cdot\left(\sum_{j=1}^{N} \max_{(a_t,\theta_t)_{t\geq 1}} \sum_{t=1}^{\infty} \delta^{t-1}\left|u^j\left(a_t, \theta_t\right)\right|\right),
\end{aligned}$$

where $L^i = \max_{(a_t,\theta_t)_{t=1}^{\tilde{T}}} \sum_{t=1}^{\tilde{T}} \delta^{t-1} \left| u^i\left(a_t, \theta_t\right) - p_t^i\left(\theta_t^{-i}; a_{t-1}, \theta_{t-1}^{-i}\right)\right| < \infty$. That is, there is a uniform upper bound on agent i 's realized discounted payoff under transfers $\{p_t^i\}_{t\geq 1}$.

Main Results

- Assume all agents other than i report their signals truthfully and focus on agent i 's incentive problem.
- Fix a socially efficient allocation rule a_t^*. By the one-shot deviation principle, we only need to show that after any public history up to period t, agent i does not benefit from deviating to $r_t^i \neq \theta_t^i$ and $r_s^i = \theta_s^i$ for $s > t$.
- If agent i reports truthfully in period t, i.e., $r_t^i = \theta_t^i$, her continuation payoff is

$$
\begin{aligned}
& u^i\left(a_t^*\left(\theta_t\right), \theta_t\right) - p_t^i\left(\theta_t^{-i}, r_{t-1}^i; a_{t-1}, \theta_{t-1}^i\right) \\
& + \delta \sum_{\theta_{t+1} \in \Theta_{t+1}} \left[W\left(\theta_{t+1}\right) - p_{t+1}^i\left(\theta_{t+1}^{-i}, \theta_t^i; a_t^*\left(\theta_t\right), \theta_t^{-i}\right)\right] \mu\left(\theta_{t+1} \mid a_t^*\left(\theta_t\right), \theta_t\right) \\
& = u^i\left(a_t^*\left(\theta_t\right), \theta_t\right) + \sum_{j \neq i} u^j\left(a_t^*\left(\theta_t\right), \theta_t\right) - p_t^i\left(\theta_t^{-i}, r_{t-1}^i; a_{t-1}, \theta_{t-1}^i\right) \\
& + \delta \sum_{\theta_{t+1} \in \Theta_{t+1}} W\left(\theta_{t+1}\right) \mu\left(\theta_{t+1} \mid a_t^*\left(\theta_t\right), \theta_t\right) \\
& = W\left(\theta_t\right) - p_t^i\left(\theta_t^{-i}, r_{t-1}^i; a_{t-1}, \theta_{t-1}^i\right)
\end{aligned}
$$

Main Results

- Suppose agent i deviates to a message r_t^i such that $a_t^*\left(r_t^i, \theta_t^{-i}\right) = a_t^*\left(\theta_t\right)$. Then her continuation payoff satisfies

$$
\begin{aligned}
& u^i\left(a_t^*\left(r_t^i, \theta_t^{-i}\right), \theta_t\right) - p_t^i\left(\theta_t^{-i}, r_{t-1}^i; a_{t-1}, \theta_{t-1}^i\right) \\
& + \delta \sum_{\theta_{t+1} \in \Theta_{t+1}} \left[W\left(\theta_{t+1}\right) - p_{t+1}^i\left(\theta_{t+1}^{-i}, r_t^i; a_t^*\left(r_t^i, \theta_t^{-i}\right), \theta_t^{-i}\right)\right] \mu\left(\theta_{t+1} \mid a_t^*\left(r_t^i, \theta_t^{-i}\right), \theta_t\right) \\
& = u^i\left(a_t^*\left(\theta_t\right), \theta_t\right) - p_t^i\left(\theta_t^{-i}, r_{t-1}^i; a_{t-1}, \theta_{t-1}^i\right) \\
& + \delta \sum_{\theta_{t+1} \in \Theta_{t+1}} \left[W\left(\theta_{t+1}\right) - p_{t+1}^i\left(\theta_{t+1}^{-i}, r_t^i; a_t^*\left(\theta_t\right), \theta_t^{-i}\right)\right] \mu\left(\theta_{t+1} \mid a_t^*\left(\theta_t\right), \theta_t\right) \\
& \leq u^i\left(a_t^*\left(\theta_t\right), \theta_t\right) - p_t^i\left(\theta_t^{-i}, r_{t-1}^i; a_{t-1}, \theta_{t-1}^i\right) \\
& + \delta \sum_{\theta_{t+1} \in \Theta_{t+1}} \left[W\left(\theta_{t+1}\right) - p_{t+1}^i\left(\theta_{t+1}^{-i}, \theta_t^i; a_t^*\left(\theta_t\right), \theta_t^{-i}\right)\right] \mu\left(\theta_{t+1} \mid a_t^*\left(\theta_t\right), \theta_t\right) \\
& = W\left(\theta_t\right) - p_t^i\left(\theta_t^{-i}, r_{t-1}^i; a_{t-1}, \theta_{t-1}^i\right)
\end{aligned}
$$

Main Results

- Finally, if agent i deviates to a message r_t^i such that $a_t^*\left(r_t^i, \theta_t^{-I}\right) = a' \neq a_t^*\left(\theta_t\right)$, then her continuation payoff satisfies

$$
\begin{aligned}
& u^i\left(a_t^*\left(r_t^i, \theta_t^{-i}\right), \theta_t\right) - p_t^i\left(\theta_t^{-i}, r_{t-1}^i; a_{t-1}, \theta_{t-1}^i\right) \\
& + \delta \sum_{\theta_{t+1} \in \Theta_{t+1}} \left[W\left(\theta_{t+1}\right) - p_{t+1}^i\left(\theta_{t+1}^{-i}, r_t^i; a_t^*\left(r_t^i, \theta_t^{-i}\right), \theta_t^{-i}\right)\right] \mu\left(\theta_{t+1} \mid a_t^*\left(r_t^i, \theta_t^{-i}\right), \theta_t\right) \\
& = u^i\left(a', \theta_t\right) - p_t^i\left(\theta_t^{-i}, r_{t-1}^i; a_{t-1}, \theta_{t-1}^i\right) \\
& + \delta \sum_{\theta_{t+1} \in \Theta_{t+1}} \left[W\left(\theta_{t+1}\right) - p_{t+1}^i\left(\theta_{t+1}^{-i}, r_t^i; a', \theta_t^{-i}\right)\right] \mu\left(\theta_{t+1} \mid a', \theta_t\right) \\
& \leq u^i\left(a', \theta_t\right) - p_t^i\left(\theta_t^{-i}, r_{t-1}^i; a_{t-1}, \theta_{t-1}^i\right) \\
& + \delta \sum_{\theta_{t+1} \in \Theta_{t+1}} \left[W\left(\theta_{t+1}\right) - p_{t+1}^i\left(\theta_{t+1}^{-i}, \theta_t^i; a', \theta_t^{-i}\right)\right] \mu\left(\theta_{t+1} \mid a', \theta_t\right) \\
& = u^i\left(a', \theta_t\right) + \sum_{j \neq i} u^j\left(a', \theta_t\right) - p_t^i\left(\theta_t^{-i}, r_{t-1}^i; a_{t-1}, \theta_{t-1}^i\right) \\
& + \delta \sum_{\theta_{t+1} \in \Theta_{t+1}} W\left(\theta_{t+1}\right) \mu\left(\theta_{t+1} \mid a', \theta_t\right) \\
& \leq W\left(\theta_t\right) - p_t^i\left(\theta_t^{-i}, r_{t-1}^i; a_{t-1}, \theta_{t-1}^i\right)
\end{aligned}
$$

Main Results

Lemma (A.4)

Under Assumptions 1 and 3, for each i and $t < T$, there exists a transfer function $p_{t+1}^{i} : \Theta_{t+1}^{-i} \times A_t \times \Theta_t^{-i} \to \mathbb{R}_+$ such that

$$-\sum_{j \neq i} u^j \left(a_t, \theta_t^i, \theta_t^{-i}\right) = \delta \sum_{\theta_{t+1}^{-i} \in \Theta_{t+1}^{-i}} p_{t+1}^{i} \left(\theta_{t+1}^{-i}; a_t, \theta_t^{-i}\right) \mu_{t+1}^{-i} \left(\theta_{t+1}^{-i} \mid a_t, \theta_t\right)$$

for every a_t, θ_t^{-i} and $\theta_t^i \in \Theta_t^i$.

- Proof. Fix any a_t and θ_t^{-i}. Equality is a system of linear equations. Since the transition matrix $\mu_{t+1}^{-i} \left(\theta_{t+1}^{-i} \mid a_t, \theta_t^i, \theta_t^{-i}\right)$ from θ_t^i to θ_{t+1}^{-i} has full rank under Assumption 3, the system of equations has a solution given by

$$\mathbf{p}_{t+1}^{i} \left(\cdot; a_t, \theta_t^{-i}\right) = \frac{1}{\delta} \left(M_{t+1}^{-i} \left(a_t, \theta_t^{-i}\right)\right)^{+} \mathbf{u}^{-i} \left(\cdot; a_t, \theta_t^{-i}\right)$$

where $\mathbf{p}_{t+1}^{i} \left(\cdot; a_t, \theta_t^{-i}\right) = \left(p_{t+1}^{i} \left(\theta_{t+1}^{-i}; a_t, \theta_t^{-i}\right)\right)_{\theta_{t+1}^{-i}}$ and $\mathbf{u}^{-i} \left(\cdot; a_t, \theta_t^{-i}\right) = \left(-\sum_{j \neq i} u^j \left(a_t, \theta_t^i, \theta_t^{-i}\right)\right)_{\theta_t^i}$ are column vectors.

Main Results

- The next result shows that under a slightly stronger condition on the transition probabilities, the dynamic efficient allocations are incentive compatible with a sequence of "VCG-type" transfers for each agent in the sense that each agent's report in each period affects her payoff only through the determination of allocation.

Theorem (3.2)

Under Assumptions 1, 3, and 4, there exists a sequence of transfers

$$\bar{p}^i_{t+1} : \Theta^{-i}_{t+1} \times A_t \times \Theta^{-i}_t \to \mathbb{R} \quad \forall i, t < T$$

such that the efficient dynamic mechanism $\{a^*_t, \bar{p}_t\}$ is periodic ex post incentive compatible.

Corollary (3.3)

In the finite-horizon case ($T < \infty$), under Assumptions 1 and 2, there exists a sequence of transfers

$$\bar{p}^i_{t+1} : \Theta^{-i}_{t+1} \times A_t \times \Theta^{-i}_t \to \mathbb{R} \quad \forall i, t < T$$

such that the (almost efficient) dynamic mechanism $\left\{(a^*_t, \bar{p}_t)_{t<T}, \bar{a}_T\right\}$, where, for all θ_T, $\bar{a}_T(\theta_T) \equiv \bar{a}$ for some $\bar{a} \in A_T$, is periodic ex post incentive compatible.

Indirect implementation with auctions: An example

- In the previous section, we focused on direct dynamic mechanisms to address feasibility issues: the existence of efficient dynamic mechanisms that are periodic ex post incentive compatible.
- A natural question is whether there are indirect mechanisms, such as auctions, that implement the direct mechanisms.
- One difficulty of this is that the history-dependent transfers in our direct mechanisms are complex in general.
- Nevertheless, the VCG aspect of the direct mechanisms suggests a natural way for indirect implementation: static auctions combined with contingent transfers.
- Here we present a repeated allocation problem in which no static auction format is efficient, but history-dependent transfers facilitate implementing our efficient direct mechanisms with familiar auction formats. In every period $t = 1, 2, \ldots, \infty$, an indivisible object is to be allocated to a bidder $i \in \{1, 2, \ldots, N\}$.
- The allocation $a_t \in \{1, 2, \ldots, N\}$ determines which bidder gets the object in period t. We assume that bidder i's valuation of the object in period t is symmetric and given by

$$v^i(\theta_t) = \theta_t^i + \gamma \sum_{j \neq i} \theta_t^j$$

where $\gamma > 0$ is a measure of interdependence in valuations.

Indirect implementation with auctions: An example

- We also assume that the allocation does not affect the evolution of agents' private information. This implies that it is efficient to allocate each object to the agent (with an arbitrary tie-breaking rule) whose valuation of the object is the highest. Finally, we assume that for each i, t, and θ_t, there exists a map $\eta^{-i} : \Theta_{t+1}^{-i} \to \mathbb{R}$ such that

$$\frac{1}{N} \sum_{j} \theta_t^j = \sum_{\theta_{t+1}^{-i}} \eta^{-i} \left(\theta_{t+1}^{-i} \right) \mu_{t+1}^{-i} \left(\theta_{t+1}^{-i} \mid \theta_t \right)$$

- Condition (2), which is stronger than Assumption 3, states that the average of all bidders' private signals today is an unbiased estimation of an index that aggregates all but one bidder's signal tomorrow.

Indirect implementation with auctions: An example

- First note that when $\gamma \in (0,1]$, the standard single-crossing condition on valuations is satisfied; this ensures that the symmetric equilibrium of a repeated sealed-bid second price auction is efficient. Alternatively, when $\gamma > 1$, it is well known that no standard auction format is efficient.
- Applying the insight from the direct mechanisms with history-dependent transfers, we consider the following dynamic winner-pay auction format: Step 1. Bidder i submits a sealed bid $b_t^i \in \mathbb{R}$ in period t. Step 2. The object is then allocated to the bidder who submitted the lowest bid (with an arbitrary tie-breaking rule),

$$a_t\left(b_t^1, \ldots, b_t^N\right) = \min\left\{i \in \{1, \ldots, N\} : b_t^i \leq b_t^j, \forall j \neq i\right\}$$

- Step 3. The winner in period t pays the second lowest bid in this period; other bidders does not pay. Step 4. The winner also pays a contingent transfer in period $t+1$ that depends on all other bidders' bids in both period t and $t+1$. Formally, if bidder i wins in period t, he pays $b_t^j = \min\left\{b_t^k : k \neq i\right\}$ in period t and r_{t+1}^i in period $t+1$, which is given by

$$r_{t+1}^i\left(b_{t+1}^{-i}, b_t^{-i}\right) = \frac{N\gamma}{\delta}\left[\eta^{-i}\left(\frac{b_{t+1}^{-i}}{1+\gamma(N-1)}\right) - \frac{b_t^j}{1+\gamma(N-1)}\right]$$

- It is straightforward to verify that a symmetric and monotone equilibrium in the constructed auction is, for all i and t, $b_t^i\left(\theta_t^i\right) = (1+\gamma(N-1))\theta_t^i$. Moreover, this symmetric strategy profile remains an equilibrium of the dynamic auction irrespective of the bids or winners that the auctioneer may choose to disclose to some bidders.

Infinite signal spaces

- In this section, we study the case where agents' signal spaces are infinite and focus on the infinite-horizon setting ($T = \infty$).
- We first identify conditions on the transition probabilities under which there exist mechanisms that are approximately periodic ex-post incentive compatible, thereby establishing infinite-signal versions of Theorems 3.1 and 3.2 under a weaker solution concept.
- We then show that under stronger conditions there are mechanisms that are periodic ex post incentive compatible.
- Suppose for each i and t, Θ_t^i is the unit interval $[0,1]$ endowed with the Borel sigma algebra, $A_t = A$, where A is a finite set, and $u^i(a_t, \cdot)$ is continuous in θ_t for each $a_t \in A$.
- In addition, we assume that the transition probability $\mu(\theta_{t+1} \mid a_t, \theta_t)$ is stationary (independent of t) and has a continuous density representation $f(\theta_{t+1} \mid a_t, \theta_t)$. The marginal density on Θ_{t+1}^{-i} is denoted by $f^{-i}\left(\theta_{t+1}^{-i} \mid a_t, \theta_t\right)$.

Approximate periodic ex-post incentive compatibility

- First consider a weakening of periodic ex post equilibrium, which requires that after any history, truth-telling is "almost" a best response if all other agents report truthfully. Formally, for any $\varepsilon > 0$, we say that the mechanism $\{a_t^*, p_t\}_{t\geq 1}$ is ε-periodic ex-post incentive compatible if for each t, i, h_t^i, and θ_t^i,

$$u^i\left(a_t^*\left(\theta_t^i, \theta_t^{-i}\right), \theta_t\right) - p_t^i\left(h_t, \theta_t^i, \theta_t^{-i}\right) + \delta\mathbb{E}\left[V^i\left(h_{t+1}^i\right) \mid a_t^*\left(\theta_t^i, \theta_t^{-i}\right), \theta_t\right]$$
$$\geq u^i\left(a_t^*\left(r_t^i, \theta_t^{-i}\right), \theta_t\right) - p_t^i\left(h_t, r_t^i, \theta_t^{-i}\right) + \delta\mathbb{E}\left[V^i\left(h_{t+1}^i\right) \mid a_t^*\left(r_t^i, \theta_t^{-i}\right), \theta_t\right] - \varepsilon$$

  for any $r_t^i \in \Theta_t^i$, where $V^i\left(h_{t+1}^i\right)$ is the continuation payoff of agent i if all agent report truthfully from period $t+1$ onward.
- The condition implies that after any history, any truthfully from period $t+1$ onward. The condition implies that after any history, any one-shot deviation from truth-telling would yield an agent at most ε improvement in his continuation payoff.
- Note that because of discounting, if a mechanism is ε-periodic ex-post incentive compatible, then truth-telling consists of a (contemporaneous) $\varepsilon(1-\ \delta)^{-1}$-perfect ex-post equilibrium.

Approximate periodic ex-post incentive compatibility

- In the following two lemmas, we identify conditions on the transition densities $f^{-i}\left(\theta_{t+1}^{-i} \mid a_t, \theta_t\right)$ such that for every $\varepsilon > 0$, there exist transfer schedules p_t that are ε periodic ex post incentive compatible.

Lemma (5.1)

Fix any i, t, a_t, and θ_t^{-i}. Iffor every θ_t^i and every $\mu^i \in \Delta\left(\Theta_t^i\right)$,

$$f^{-i}\left(\cdot \mid a_t, \theta_t^i, \theta_t^{-i}\right) = \int_{\Theta_t^i} f^{-i}\left(\cdot \mid a_t, \tilde{\theta}_t^i, \theta_t^{-i}\right) \mu^i\left(d\tilde{\theta}_t^i\right) \quad \Rightarrow \quad \mu^i\left(\left\{\theta_t^i\right\}\right) = 1$$

then for any $\varepsilon > 0$, there exist transfers that are $p_{t+1}^i\left(\theta_{t+1}^{-i}, \theta_t^i; a_t, \theta_t^{-i}\right)$ measurable in θ_t^i and continuous in θ_{t+1}^{-i} and θ_t^{-i} such that

$$\max_{\theta_t^i \in \Theta_t^i} \left| -\sum_{j \neq i} u^j\left(a_t, \theta_t\right) - \delta \int_{\Theta_{t+1}^{-i}} p_{t+1}^i\left(\theta_{t+1}^{-i}, \theta_t^i; a_t, \theta_t^{-i}\right) f^{-i}\left(\theta_{t+1}^{-i} \mid a_t, \theta_t\right) d\theta_{t+1}^{-i} \right| \leq \varepsilon$$

and

$$\begin{aligned} &\int_{\Theta_{t+1}^{-i}} p_{t+1}^i\left(\theta_{t+1}^{-i}, \theta_t^i; a_t, \theta_t^{-i}\right) f^{-i}\left(\theta_{t+1}^{-i} \mid a_t, \theta_t\right) d\theta_{t+1}^{-i} \\ &\leq \int_{\Theta_{t+1}^{-i}} p_{t+1}^i\left(\theta_{t+1}^{-i}, r_t^i; a_t, \theta_t^{-i}\right) f^{-i}\left(\theta_{t+1}^{-i} \mid a_t, \theta_t\right) d\theta_{t+1}^{-i} \end{aligned}$$

for any $r_t^i \in \Theta_t^i$.

Periodic ex post incentive compatibility

- The lemmas in Section 5.1 generalize the main results in Section 3. However, they are not very satisfactory, especially in the dynamic environments.
- That is, agents may well deviate from truth-telling under ε-periodic ex post incentive compatibility, yet they evaluate their continuation payoffs assuming others are always truthful.
- In this section, we strengthen the results to (full) periodic ex post incentive compatibility under stronger correlation conditions.
- Note that the contingent transfers that deliver ε-periodic ex post incentive compatibility in Section 5.1 depend on the reports one period ahead, whereas in principle they could depend on reports in the more distant future. Therefore, we consider the contingent transfers

$$p_t^i : \Theta_t^i \times \Theta_t^{-i} \times A_t \times \prod_{\tau > t} \left(\Theta_\tau^{-i} \times A_\tau \right) \to \mathbb{R}.$$

- Intuitively, if agent i 's current private signal θ_t^i is correlated with other agents' future signals $\left\{\theta_\tau^{-i}\right\}_{\tau>t}$, then provided that other agents always report truthfully, it is possible to use the entire sequence, $\left\{\theta_\tau^{-i}\right\}_{\tau>t}$, to provide incentive for agent i to report θ_t^i truthfully.
- We might fill the gap in ε-incentive compatibility with an infinite sequence of correlated signals. We formalize this intuition in the next two propositions.

Periodic ex post incentive compatibility

- For each i, t, and $\tau > t$, let $f_\tau^{-i}\left(\theta_\tau^{-i} \mid a_t, \ldots, a_{\tau-1}, \theta_t\right)$ denote the marginal density on Θ_τ^{-i} given any $a_t, \ldots, a_\tau$ and θ_t.

Proposition (5.3)

Fix any i, t, and θ_t^{-i}. If for every $\tau > t$, $(a_t, \ldots, a_\tau) \in A_t \times \cdots \times A_\tau$, $\theta_t^i \in \Theta_t^i$, and $\mu_\tau^i \in \Delta\left(\Theta_t^i\right)$,

$$
\begin{aligned}
& f_\tau^{-i}\left(\cdot \mid a_t, \ldots, a_{\tau-1}, \theta_t^i, \theta_t^{-i}\right) = \int_{\theta_t^i} f^{-i}\left(\cdot \mid a_t, \ldots, a_{\tau-1}, \tilde{\theta}_t^i, \theta_t^{-i}\right) \mu^i\left(d\tilde{\theta}_t^i\right) \\
\Rightarrow \quad & \mu_\tau^i\left(\left\{\theta_t^i\right\}\right) = 1
\end{aligned}
$$

then there exists a sequence of transfers $\left(p_\tau^i\left(\theta_\tau^{-i}, \theta_t^i; a_t, \ldots, a_{\tau-1}, \theta_t^{-i}\right)\right)_{\tau > t}$ measurable in θ_t^i and continuous in θ_τ^{-i} and θ_t^{-i} such that

$$
-\sum_{j \neq i} u^j\left(a_t, \theta_t\right) = \sum_{\tau=t+1}^{\infty} \delta^{\tau-t} \int_{\Theta_\tau^{-i}} p_\tau^i\left(\theta_\tau^{-i}, \theta_t^i; a_t, \ldots, a_{\tau-1}, \theta_t^{-i}\right) f_\tau^{-i}\left(\theta_\tau^{-i} \mid a_t, \ldots, a_{\tau-1}, \theta_t\right) d\theta_\tau^{-i}
$$

and

$$
\begin{aligned}
& \int_{\Theta_\tau^{-i}} p_\tau^i\left(\theta_\tau^{-i}, \theta_t^i; a_t, \ldots, a_{\tau-1}, \theta_t^{-i}\right) f_\tau^{-i}\left(\theta_\tau^{-i} \mid a_t, \ldots, a_{\tau-1}, \theta_t\right) d\theta_\tau^{-i} \\
& \quad \leq \int_{\Theta_\tau^{-i}} p_\tau^i\left(\theta_\tau^{-i}, r_t^i; a_t, \ldots, a_{\tau-1}, \theta_t^{-i}\right) f_\tau^{-i}\left(\theta_\tau^{-i} \mid a_t, \ldots, a_{\tau-1}, \theta_t\right) d\theta_\tau^{-i}
\end{aligned}
$$

for any $r_t^i \in \Theta_t^i$ and $\tau > t$.

Periodic ex post incentive compatibility

- For each i, t, and $\tau > t$, let $f_\tau^{-i}\left(\theta_\tau^{-i} \mid a_t, \ldots, a_{\tau-1}, \theta_t\right)$ denote the marginal density on Θ_τ^{-i} given any $a_t, \ldots, a_\tau$ and θ_t.

Proposition (5.4)

Fix any i, t, and θ_t^{-i}. If for every $\tau > t, (a_t, \ldots, a_\tau) \in A_t \times \cdots \times A_\tau$, there does not exist a nonzero signed measure η_τ^i on the Borel subsets of Θ_t^i such that

$$\int_{\Theta_t^i} f_\tau^{-i}\left(\cdot \mid a_t, \ldots, a_{\tau-1}, \tilde{\theta}_t^i, \theta_t^{-i}\right) \eta_\tau^i\left(d\tilde{\theta}_t^i\right) = 0$$

then there exists a sequence of transfers $\left(p_\tau^i\left(\theta_\tau^{-i}, \theta_t^i; a_t, \ldots, a_{\tau-1}, \theta_t^{-i}\right)\right)_{\tau>t}$ measurable in θ_t^i and continuous in θ_τ^{-i} and θ_t^{-i} such that

$$\begin{aligned} & -\sum_{j \neq i} u^j\left(a_t, \theta_t\right) \\ & = \sum_{\tau=t+1}^{\infty} \delta^{\tau-t} \int_{\Theta_\tau^{-i}} p_\tau^i\left(\theta_\tau^{-i}, \theta_t^i; a_t, \ldots, a_{\tau-1}, \theta_t^{-i}\right) f_\tau^{-i}\left(\theta_\tau^{-i} \mid a_t, \ldots, a_{\tau-1}, \theta_t\right) d\theta_\tau^{-i} \end{aligned}$$

Conclusion

- Dynamic mechanism design features a richer family of history-dependent transfers compared with the static counterpart.
- This paper has taken a first step toward understanding the implications of such richness on efficient implementations in general environments with interdependent valuations.
- In particular, we have shown how intertemporal correlation of private information leads to contingent transfers that resemble dynamic VCG mechanisms.
- We also emphasize that while the theoretical possibility results in this paper serve as a benchmark for the design of efficient mechanisms, the practicality of contingent transfers may vary with specific economic problems.

Thanks!